Approximation Theorems for Positive Operators on L^p-Spaces

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Let $1 < r \leqslant p < \infty$. Approximation theorems for positive contractions in $\mathscr{L}(L^p(m), L^r(n))$ are presented. The characterization of norm attaining extreme positive contractions is given. Note that these extreme operators are also exposed points of the positive part of the unit ball of $\mathscr{L}(l^p, l^r)$. © 1990 Academic Press, Inc.

In this paper, we investigate the set of positive contractions in $\mathscr{L}(L^p, L^r)$. Results presented here have the same character as the Krein–Milman theorem. We approximate positive contractions on L^p by convex combinations of norm attaining extreme positive contractions (Theorem 1). J. Hennefeld [17] proved that the unit ball of the space of compact operators on l^p is the norm closed convex hull of its extreme points for $1 \leqslant p < \infty$, $p \neq 2$. In Section 2, we present approximation theorems for positive contractions. Using Lindenstrauss's result, we show that a certain class of norm attaining operators is norm·dense. Next we prove that the positive part of the unit ball of operators is the closure on the convex hull of the set of norm attaining extreme positive contractions in the strong operator topology. The approximation theorems for operators on l^∞ and $L^\infty[0, 1]$ are presented in [23, 21, 22]. Note that the convex hull of positive invertible isometries of $L^p[0, 1]$ is strong operator dense in the set of positive contractions acting on $L^p[0, 1]$ (Corollary 1). This is related to a result of J. R. Brown [3].

In Section 4 we give a characterization of the norm attaining extreme positive contractions in several cases. Note that this is related to the characterization of extreme doubly stochastic measures. From the facts presented in [16] it follows that the locally affine structure of the positive contractions in $\mathscr{L}(L^p(m), L^r(n))$ $(1 < r \leqslant p < \infty)$ is similar to the structure of doubly stochastic measures. Therefore in Section 3 we present those properties of doubly stochastic matrices which we use in Section 4 for the description of norm attaining extreme positive contractions in

$\mathscr{L}(L^p(m), L^r(n))$. In the finite dimensional case, extreme positive contractions are characterized in [13]. We also prove that the norm attaining extreme positive contractions are exposed points of the positive part of the unit ball of operators.

1. TERMINOLOGY AND NOTATION

Let $(X, \mathscr{A}, m)$ be a σ-finite measure space. As usual, we denote by $L^p(m)$, $1 \leqslant p < \infty$, the Banach lattice of all p-summable real-valued functions on X with standard norm and order. If $X = \{1, 2, ..., n\}$, $n > \infty$ $(X = \mathbb{N})$, and m is a counting measure we write l_n^p (l^p) instead of $L^p(m)$. If $X = [0, 1]$ and m is the Lebesgue measure, we write briefly $L^p[0, 1]$. In the case of sequence l^p-spaces, we denote by $\{e_k\}$ the canonical base (i.e., $(e_k)_j = \delta_{kj}$). The cone of positive functions ($f \geqslant 0$) in $L^p(m)$ is denoted by $L_+^p(m)$. The adjoint space $[L^p(m)]'$ is identified with $L^{p'}(m)$, where $1/p + 1/p' = 1$. For $f \in L^p(m)$, we define its support by $\operatorname{supp} f = \{x \in X : f(x) \neq 0\}$. Note that $\operatorname{supp} f$ is defined modulo null sets.

Let $1 \leqslant r < \infty$ and let $(Y, \mathscr{B}, n)$ be a σ-finite measure space. We denote by $\mathscr{L}(L^p(m), L^r(n))$ the Banach space of all linear bounded operators from $L^p(m)$ into $L^r(n)$. An operator is said to be positive ($T \geqslant 0$) if $Tf \geqslant 0$ whenever $f \geqslant 0$. The set of all positive operators is denoted by $\mathscr{L}_+(L^p(m), L^r(n))$. We denote the positive part of the unit ball of $\mathscr{L}_+(L^p(m), L^r(n))$ by $\mathscr{P}$. We define the support of a positive operator T, denoted by $\operatorname{supp} T$, as a maximal set $A \subset X$ such that $T1_{A^c} = 0$. We say that an operator $T \in \mathscr{L}_+(L^p(m), L^r(n))$ is elementary provided there are no non-zero operators $T_1, T_2 \in \mathscr{L}_+(L^p(m), L^r(n))$ such that $T = T_1 + T_2$ and $(\operatorname{supp} T_1) \cap (\operatorname{supp} T_2) = (\operatorname{supp} T_1^*) \cap (\operatorname{supp} T_2^*) = \varnothing$ (cf. [16]).

For $f \in L^{p'}(m)$, $g \in L^r(n)$ we denote by $g \otimes f$ the operator from $\mathscr{L}(L^p(m), L^r(n))$ such that $(g \otimes f)(h) = g \int fh\, dm$. Note that if $\operatorname{supp} f = X$, $\operatorname{supp} g = Y$ then $g \otimes f$ is an elementary operator. Also if $T \geqslant S \geqslant 0$, and S is elementary, then T is an elementary operator. For $T \in \mathscr{L}(L^p(m), L^r(n))$, we define its isometric domain as $M(T) = \{f \in L^p(m) : \|Tf\| = \|T\| \|f\|\}$. If $M(T) \neq \{0\}$, then we say that T is norm attaining. The properties of $M(T)$ and $J(T) = \{\operatorname{supp} f : f \in M(T)\}$ are considered in [16].

To every operator $T \in \mathscr{L}(l^p, l^r)$ (or $\mathscr{L}(l_n^p, l_n^r)$), there corresponds a unique matrix (t_{ij}) with real entries, such that $(Tf)_i = \sum_j t_{ij} f_j$. Clearly the adjoint operator $T^* \in \mathscr{L}(l^{r'}, l^{p'})$ with $1/p + 1/p' = 1/r + 1/r' = 1$ is determined in the same manner by the transposed matrix. We identify an operator with its matrix.

The graph $G(T)$ of a matrix $T = (t_{ij})$ is defined by the following formula. To the ith row, there corresponds a (row) node x_i, $i = 1, 2, ...$, and to the jth column, there corresponds a column node y_j, $j = 1, 2, ...$. There is an edge joining x_i and y_j if and only if $t_{ij} \neq 0$.

2. APPROXIMATION THEOREMS

The proposition below is a modification of Theorem 1 in [26].

PROPOSITION 1. *Let $1 < r \leqslant p < \infty$ and let*

$$\mathscr{N} = \{ T \in \mathscr{L}_+(L^p(m), L^r(n)); X \in J(T), Y \in J(T^*) \}.$$

Then $\mathscr{N}$ is norm dense in $\mathscr{L}_+(L^p(m), L^r(n))$.

Proof. Let us fix $\varepsilon > 0$ and $T_0 \in \mathscr{L}_+(L^p(m), L^r(n))$. Let $f \in L_+^p(m)$, $g \in L_+^r(n)$ where $\|f\| = \|g\| = 1$, $\operatorname{supp} f = X$, $\operatorname{supp} g = Y$. Put $T = T_0 + \varepsilon g \otimes f$. Obviously $\|T - T_0\| = \varepsilon$. Now we construct an operator $\hat{T} \in \mathscr{N}$ such that $\|\hat{T} - T\| < \varepsilon$. We use the construction given in the proof of Theorem 1 in [26]. Assume without loss of generality that $\|T\| = 1$, $0 < \varepsilon < 1/3$. We choose first a monotonically decreasing sequence $\{\varepsilon_k\}$ of positive numbers such that

$$2 \sum_{i=1}^{\infty} \varepsilon_i < \varepsilon, \qquad 2 \sum_{i=k-1}^{\infty} \varepsilon_i < \varepsilon_k^2, \, \varepsilon_k < 1/10k,$$

$k = 1, 2, \ldots$. We next choose inductively sequences $\{T_k\}$, $\{f_k\}$, $\{g_k\}$ such that $T_k \in \mathscr{L}(L^p(m), L^r(n))$, $f_k \in L_+^p(m)$, $g_k \in L_+^r(n)$ satisfying

$$T_1 = T$$

$$\|T_k f_k\| \geqslant \|T_k\| - \varepsilon_k^2, \qquad \|f_k\| = 1, \qquad k = 1, 2, \ldots$$

$$g_k(T_k f_k) = \|T_k f_k\|, \qquad \|g_k\| = 1, \qquad k = 1, 2, \ldots$$

$$T_{k+1} h = T_k h + \varepsilon_k g_k(T_k h) T_k f_k, \qquad h \in L^p(m), \qquad k = 1, 2, \ldots.$$

The sequence $\{T_k\}$ converges in norm to an operator $\hat{T}$ satisfying $\|\hat{T} - T\| < \varepsilon$ and $\hat{T}$ is a norm attaining operator (see the proof of Theorem 1 in [26].)

Note that $T_{k+1} \geqslant T_k$, $k = 1, 2, \ldots$. Thus $\hat{T} = \lim T_k \geqslant T \geqslant \varepsilon g \otimes f$. We have $\operatorname{supp} \hat{T} = \operatorname{supp} f = X$, and $\operatorname{supp} \hat{T}^* = \operatorname{supp} g = Y$. Since the operator T is an elementary operator, by Theorem 4 and 5 in [16], $\hat{T}$ attains its norm at a function whose support is a whole space X, i.e., $X \in J(\hat{T})$. Similarly $Y \in J(T^*)$. Thus $\hat{T} \in \mathscr{N}$.

Therefore there exists $\hat{T} \in \mathscr{N}$ such that $\|\hat{T} - T_0\| < 2\varepsilon$.

By Theorem 2 in [12], it follows that the extreme positive contractions T, which attain their norm at a function f with $\operatorname{supp} f = \operatorname{supp} T$, are related to the extreme doubly stochastic operators. The theorem below says that the set of such a class of extreme positive contractions is large enough. Really the set $\mathscr{N} \cap \operatorname{ext} \mathscr{P}$ is smaller than this class of extreme operators. Note that the zero operator belongs to $\mathscr{N} \cap \operatorname{ext} \mathscr{P}$.

For $f \in L_+^p(m)$, $g \in L^r(n)$, we define

$$\mathscr{A}_{f,g} = \{T \in \mathscr{L}_+(L^p(m), L^r(n)): Tf = g, T^*g^{r-1} = f^{p-1}, \text{supp } T = \text{supp } f\}.$$

THEOREM 1. *Let* $1 < r \leqslant p < \infty$. *Then the convex hull*

$$\text{conv}(\mathscr{N} \cap \text{ext } \mathscr{P})$$

is strong operator dense in the positive part of the unit sphere of $\mathscr{L}(L^p(m), L^r(n))$.

Proof. By Proposition 1, it is sufficient to show that for every $f \in L_+^p(m)$, $g \in L_+^r(n)$, with $\|f\| = \|g\| = 1$ the convex hull of ext $\mathscr{A}_{f,g}$ ($\subset \mathscr{N}^+ \cap \text{ext } \mathscr{P}$) is strong operator dense in the convex set $\mathscr{A}_{f,g}$. Since convex sets have the same closure in the weak operator and strong operator topologies [10, p. 447] all we need to show is that $\mathscr{A}_{f,g}$ is compact in the weak operator topology. Let T belong to the closure in the weak operator topology of $\mathscr{A}_{f,g}$ and let the net $T_\alpha \in \mathscr{A}_{f,g}$ converge to T. The condition $1 = \langle g^{r-1}, z_\alpha f \rangle \to \langle g^{r-1}, Tf \rangle$ implies that $\langle g^{r-1}, Tf \rangle = 1$. By the strict convexity of $L^r(n)$ we have $Tf = g$. The operator T attains its norm on f. Hence $T^*g^{r-1} = f^{p-1}$ and supp $T \supset$ supp f. For each $v \in L^r(n)$ we have $\langle v, T_\alpha 1_{(\text{supp} f)}c \rangle = 0$. Thus $\langle v, T1_{(\text{supp} f)}c \rangle = 0$ and supp $T \subset$ supp f. Therefore $T \in \mathscr{A}_{f,g}$, i.e., $\mathscr{A}_{f,g}$ is closed. The face $\mathscr{A}_{f,g}$ is compact as a closed subset of the ball (the ball is compact as a closed subset of the ball (the ball is compact in the weak operator topology, since L^p is reflexive).

PROPOSITION 2. *Let* $L^p(m)$ *or* $L^r(n)$ *be an infinite dimensional space, Then the convex hull of the positive part of the unit sphere in* $\mathscr{L}(L^p(m), L^r(n))$ *is norm dense in the set of positive contractions.*

Proof. It is sufficient to show that 0 belongs to the closure of the convex hull. Let $L^p(m)$ be infinite dimensional. We choose a sequence $\{f_n\}$ of $L_+^p(m)$ such that $\|f_n\| = 1$ and the supports are disjoint. Let $g \in L_+^r(n)$ be such that $\|g\| = 1$, and let $T_n = (1/n) \sum_{k=1}^n g \otimes f_k^{p-1} = g \otimes (1/n) \sum_{k=1}^n f_k^{p-1}$. The operators T_n are convex combinations of elements of positive parts of the unit sphere. We have $\|T_n\| = \|(1/n) \sum_{k=1}^n f_k^{p-1}\|_{p'} = \sqrt[p']{1/n}$. Thus $T_n \to 0$. In the case when $L^r(n)$ is infinite dimensional we present analogous arguments.

If $L^p(m)$ and $L^r(n)$ are finite dimensional, then the statement of Proposition 2 is not true. Indeed, then $\mathscr{L}(L^p(m), L^r(n))$ can be identified with $\mathscr{L}(l_{k_1}^p, l_{k_2}^r)$. Since all norms are equivalent on a finite dimensional space, there exist constants $c_1, c_2 > 0$ such that

$$c_1 \sum_{i,j} |t_{ij}| \leqslant \|T\| \leqslant c_2 \sum_{i,j} |t_{ij}|.$$

If $\|T\| = 1$ and $T \geqslant 0$, then $\sum_{i,j} t_{ij} \geqslant 1/c_2$. Every operator in a convex hull has a norm greater than $c_1/c_2 > 0$.

Remark 1. If $L^p(m)$ or $L^r(n)$ is infinite dimensional by Proposition 2 and Theorem 1 we obtain that the set conv($\mathcal{N} \cap$ ext $\mathscr{P}$) is strong operator dense in the set of positive contractions.

PROPOSITION 3. *Let* $1 < r \leqslant p < \infty$, *and let* $f \in L^p[0, 1]$, $g \in L^r[0, 1]$ *be functions of norm one. Then the face* $\mathscr{A}_{f,g}$ *is affinely isomorphic to the face* $\mathscr{A}_{1,1}$.

Proof. For a function $h \in L^s[0, 1]$, $1 < s < \infty$, we define a mapping $\tau_{h,s} : \text{supp } h \to [0, 1]$ by

$$\tau_{h,s}(x) = \int_0^x |h|^s \, dm.$$

The mapping $\tau_{h,s}$ is strictly increasing and onto. Hence it is invertible. Now we define operators S_1, S_2 by

$$(S_1 v)(x) = \frac{(v 1_{\text{supp } f})(\tau_{f,p}^{-1}(x))}{f(\tau_{f,p}^{-1}(x))}, \quad v \in L^p[0, 1]$$

$$(S_2 u)(x) = \begin{cases} g(x) u(\tau_{g,r}(x)), & x \in \text{supp } g \\ 0, & x \notin \text{supp } g \end{cases}$$

$u \in L^r[0, 1]$. The mapping

$$\phi(T) = S_2 T S_1$$

acts from $\mathscr{A}_{1,1}$ into $\mathscr{A}_{f,g}$. There exists ϕ^{-1} given by $\phi^{-1}(S) = R_2 S R_1$ where

$$(R_1 v)(x) = \begin{cases} f(x) v(\tau_{f,p}(x)), & x \in \text{supp } f \\ 0, & x \notin \text{supp } f \end{cases} \tag{1}$$

$v \in L^p[0, 1]$ and

$$(R_2 u)(x) = \frac{(u 1_{\text{supp } g})(\tau_{g,r}^{-1}(x))}{g(\tau_{g,r}^{-1}(x))} \tag{2}$$

$u \in L^r[0, 1]$. Indeed $S_1 R_1 = R_2 S_2 = I$.

Consider now $X = Y = [0, 1]$ with Lebesgue measure and $p = r$. An isometry on L^p is an extreme contraction. Then positive invertible isometries belong to the set $\mathcal{N}$. The set of positive invertible isometries on $L^p[0, 1]$ is a proper subset of $\mathcal{N} \cap$ ext $\mathscr{P}$.

THEOREM 2. *The set of positive invertible isometries on $L^p[0, 1]$ is weak operator dense in the set of positive contractions.*

Proof. The positive part of the unit sphere is weak operator dense in the positive part of the unit ball. Indeed, let $\{g_n\}$ be a sequence of $L^p_+[0, 1]$ such that $\|g_n\| = 1$ and $\{g_n\}$ converges in weak topology to 0. Let $f \in L^p_+[0, 1]$, with $\|f\| = 1$. The sequence $g_n \otimes f^{p-1}$ converges to 0 in the weak operator topology. Let R be a positive contraction. Let $\lambda_n \in [0, 1]$ be such that $\|R + \lambda_n g_n \otimes f^{p-1}\| = 1$. The weak operator limit of $(R + \lambda_n g_n \otimes f^{p-1})$ is R.

Operators of norm 1, which attain that norm at functions which have $[0, 1]$ as a support, are norm dense in the positive part of the sphere (Proposition 1). We need to show that the positive invertible isometries are weak operator dense in $\mathscr{A}_{f,g}$ for all $f, g \in L^p_+[0, 1]$ such that supp $f =$ supp $g = [0, 1]$ and $\|f\| = \|g\| = 1$.

Let $S \in \mathscr{A}_{f,g}$. Then $T = R_2 S R_1$ where R_1, R_2 are defined in (1), (2). The face $\mathscr{A}_{1,1}$ coincides with the set of all doubly stochastic operators. The set of positive invertible isometries generated by a measure-preserving transformation is weak operator dense in $\mathscr{A}_{1,1}$ [3, Theorem 1]. For $T \in \mathscr{A}_{1,1}$ there exists a sequence of positive invertible isometries T_n which converges to T. Operators $S_n = R_2^{-1} T_n R_1^{-1}$ are positive invertible isometries on $L^p[0, 1]$. For every $u \in L^p[0, 1]$ and $v \in L^p[0, 1]$ we have $\langle v, S_n u \rangle \to \langle v, Su \rangle$, i.e., S_n converges to S in weak operator topology.

Since convex sets have the same closure in the weak and strong operator topologies [10, p. 447] we obtain the following corollary.

COROLLARY 1. *The convex hull of positive invertible isometries of $L^p[0, 1]$ is strong operator dense in $\mathscr{P}$*

We remark that the convex hull of positive invertible isometries on $L^1[0, 1]$ is strong operator dense in the set of all stochastic operators [18, Corollary 1].

3. DOUBLY STOCHASTIC MEASURES

Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces such that $\mu(X) = \nu(Y)$. A measure λ on $(X \times Y, \mathscr{A} \otimes \mathscr{B})$ is called doubly stochastic with respect to μ and ν if its marginal distributions coincide with μ and ν, respectively (i.e., $\lambda(A \times Y) = \mu(A)$, $A \in \mathscr{A}$, $\lambda(X \times B) = \nu(B)$, $B \in \mathscr{B}$). The set of all doubly stochastic measures with respect to μ and ν is denoted by $\mathscr{D}(\mu, \nu)$.

If μ and ν are countable sums of atoms, then $\mathscr{D}(\mu, \nu)$ can be identified

with the set of all doubly stochastic matrices $P = (p_{ij})$ with respect to μ and ν (i.e., $p_{ij} \geqslant 0$, $\sum_i p_{ij} = \mu(\{j\})$, $\sum_j p_{ij} = \nu(\{i\})$).

If μ, ν are Lebesgue measures on $X = Y = [0, 1]$, then the relation

$$\lambda(A \times B) = \int_0^1 1_A P 1_B \, dt$$

determines a one-to-one correspondence between $\mathscr{D}(\mu, \nu)$ and the set of all double stochastic operators P on $L^\infty[0, 1]$ (see [3], additional information can be found in [2]). We recall that $P \in \mathscr{L}(L^\infty[0, 1], L^\infty[0, 1])$ is doubly stochastic if $P \geqslant 0$, $P1 = 1$, $P^*1 = 1$. Note that J. V. Ryff [30] presented an integral representation of doubly stochastic operators.

Properties of doubly stochastic measures, operators, and matrices have been studied by many authors. In particular the extreme points of $\mathscr{D}(\mu, \nu)$ have been characterized in certain cases. In the simples case (μ and ν are counting measures on $\{1, 2, ..., n\}$), by the well-known theorem of G. Birkhoff [5] the set of extreme points coincides with the set of all permutation matrices. The following fact is well known (see [24, 1, 8, 28]).

PROPERTY 1. *Let $X = Y = \mathbb{N}$, and let $\mu(\mathbb{N}) = \nu(\mathbb{N}) < \infty$. The matrix $P = (p_{ij}) \in \mathscr{D}(\mu, \nu)$ is extreme if and only if the connected components of the graph $G(P)$ are trees.*

The proof of the above property can be found also in [15, Corollary 1]. Note that the finiteness of the measures μ and ν in the assumption of Property 1 cannot be omitted (see [15]). We recall that a matrix $P \in \mathscr{D}(\mu, \nu)$ is said to be uniquely determined in $\mathscr{D}(\mu, \nu)$ by its graph provided there is no matrix $R \neq P$ in $\mathscr{D}(\mu, \nu)$ such that the graph $G(R)$ is a subgraph of $G(P)$. In [15], the following fact is proved.

PROPERTY 2. *Let $X = Y = \mathbb{N}$ and let $\mu(\mathbb{N}) = \nu(\mathbb{N}) < \infty$. Then the extreme points of $\mathscr{D}(\mu, \nu)$ are those matrices in $\mathscr{D}(\mu, \nu)$ which are uniquely determined in $\mathscr{D}(\mu, \nu)$ by their graph.*

In case of a finite support of μ and ν, the Properties 1 and 2 were known earlier (see [11, 7, 19, 6]),

In the continuous case (μ and ν are Lebesgue measures on $[0, 1]$), a description of extreme doubly stochastic measures was presented independently by Dougles [9] and Lindenstrauss [25]. The conjecture that extreme doubly stochastic measures on $[0, 1] \times [0, 1]$ are supported graphs of measurable maps or that extreme doubly stochastic operators are generated by invertible measure preserving transformations (as a natural generalization of permutation matrices) turns out not to be true. The geometric structure of extreme doubly stochastic measures is more complicated. V. Losert [27] presented an example of extreme doubly stochastic

measures on $[0, 1] \times [0, 1]$, whose support is the whole space. The set of operators generated by invertible measure preserving transformations is a propet subset of the set of extreme doubly stochastic operators. The convex hull of the set of operators generated by invertible measure preserving transformations is dense in the set of doubly stochastic operators in the L^2-strong operator topology (see [2]) and in the L^1-strong operator topology (see [20]). The problem of characterization of doubly stochastic measures on $[0, 1] \times [0, 1]$ in terms of measure theory remains open; however, some partial results are known (see [4, 31]).

4. EXTREME POSITIVE CONTRACTION

In this section, we consider extreme points of the positive part of the unit ball of $\mathscr{L}(l^p, l^r)$. In particular we show that the graph of the extreme positive contraction is a tree. The form of the graph of a matrix does not characterize extreme points. We also give a description of operators in $\mathscr{N} \cap \text{ext } \mathscr{P}$. Obviously, if one of the spaces L^p or L^r is finite dimensional, then each operator in $\mathscr{L}(L^p, L^r)$ is norm attaining and $\mathscr{N} \cap \text{ext } \mathscr{P} = \text{ext } \mathscr{P}$. Therefore our result is an extension of the result for finite dimensional spaces presented in [13]. Generally $\mathscr{N} \cap \text{ext } \mathscr{P} \neq \text{ext } \mathscr{P}$. Note that for $1 < r < p < \infty$ every operator in $\mathscr{L}(l^p, l^r)$ is compact [29, 5.1.2], so norm attaining. In this case $\mathscr{N}$ in Theorem 3 can be omitted.

THEOREM 3. Let $0 \neq T \in \mathscr{L}_+(l^p, l^r) \cap \mathscr{N}$, $1 < r \leqslant p < \infty$. Then $T \in \mathscr{N} \cap \text{ext } \mathscr{P}$ if and only $\|T\| = 1$ and the connected components of the graph $G(T)$ are trees.

Proof. Let $0 \neq T \in \mathscr{N} \cap \text{ext } \mathscr{P}$. Then $\|T\| = 1$, and there exists a vector $f = (f_j)$ such that $\|f\| = \|Tf\| = 1$, $f \geqslant 0$, $\text{supp } f = \text{supp } T$. Obviously, T is also an extreme point of the face

$$\mathscr{A}_{f, Tf} = \{ R \in \mathscr{L}_+(l^p, l^r) : Rf = Tf, \ R^*(Tf)^{r-1} = f^{p-1}, \ \text{supp } R = \text{supp } T \}.$$

By Theorem 2 in [13], $A_{f, Tf}$ is affinely isomorphic to $\mathscr{D}(\mu, \nu)$ where $\mu(\{j\}) = f_j^p$, $\nu(\{j\}) = (Tf)_i^r$. The measures μ, ν are countable sums of atoms. We have $\text{supp } \mu \subset \mathbb{N}$, $\text{supp } \nu \subset \mathbb{N}$, and $\mu(\mathbb{N}) = \sum_j f_j^p = \|f\|_p = \|Tf\|_r = \sum_i (Tf)_i^r = \nu(\mathbb{N}) < \infty$. Thus the matrix $P = (p_{ij})$ defined by

$$p_{ij} = (Tf)_i^{p-1} t_{ij} f_j$$

is extreme in $\mathscr{D}(\mu, v)$. The graphs $G(T)$ and $G(P)$ coincide. Thus, by Property 1, the connected components of $G(T)$ are trees.

Now assume that the connected components of the graph $G(T)$, $T \in \mathscr{N}$,

are trees and $\|T\| = 1$. Let f be such that $\|f\| = \|Tf\| = 1$, $f \geqslant 0$, supp $f = $ supp T. Then $P \in$ ext $\mathscr{D}(\mu, v)$, where μ, v, p_{ij} are defined as above. Because $\mathscr{D}(\mu, v)$ and $\mathscr{A}_{f, Tf}$ are affinely isomorphic, we obtain $T \in$ ext $\mathscr{A}_{f, Tf}$, so T is extreme.

COROLLARY 2. *Let* $T \in \mathscr{L}_+(l_n^p, l^r)$ *(or* $T \in \mathscr{L}_+(l^p, l_m^r)$*)* $1 < r \leqslant p < \infty$, *and let* $\|T\| = 1$. *Then* T *is an extreme positive contraction if and only if the connected components of the graph* $G(T)$ *are trees and* supp $T \in \mathscr{T}(T)$.

Remark 2. Let $1 < r \leqslant p < \infty$, and let $f \in l^p$, $g \in l^r$ be such that $\|f\| = \|g\| = 1$, $f > 0$, $g > 0$. Define measures μ and v of $\mathbb{N}$ by $\mu(\{j\}) = f_j^p$, $v(\{i\}) = g_i^r$. The face

$$\mathscr{A}_{f, g} = \{R \in \mathscr{L}_+(l^p, l^r): Rf = g, R^*g^{r-1} = f^{p-1}, \text{supp } R = \text{supp } f\}$$

of the unit ball of $\mathscr{L}(l^p, l^r)$ is affinely isomorphic to $\mathscr{D}(\mu, v)$ (Theorem 2 in [13]). Because the elements of ext $\mathscr{D}(\mu, v)$ are uniquely determined in $\mathscr{D}(\mu, v)$ by their graphs (Property 2), also the operator $T \in$ ext $\mathscr{A}_{f, g}$ is uniquely determined in $\mathscr{A}_{f, g}$ by its graph $G(T)$. Therefore for a positive contraction T such that $Tf = g$, and supp $f = $ supp T, if the graph $G(T)$ has no cycle, then T is uniquely determined by f, g, and $G(T)$.

THEOREM 4. *Let* $T \in \mathscr{L}_+(l_n^p, L^p[0, 1])$, $n < \infty$, $1 \leqslant p < \infty$. T *is an extreme positive contraction if and only if* $\|Te_j\| = 0$ *or* 1, $j = 1, 2, ..., n$, *and the sets* supp Te_1, supp Te_2, ..., supp Te_n *are disjoint subsets of* $[0, 1]$ *(*$\{e_j\}_{j=1}^n$ *is the canonical base of* l_n^p*)*.

Proof. We have supp $T = \{j: Te_j \neq 0\}$ and supp $T^* = \sum^n \text{supp } Te_j$. It is easy to see that each $T \in \mathscr{L}_+(l_n^p, L^p[0, 1])$ can be represented as a finite sum of some nonzero elementary operators T_s, $T = \sum_{s=1}^{s_0} T_s$ with supp T_s disjoint and supp T_s^* disjoint. We have $T \in$ ext $\mathscr{P}$ if and only if every $T_s \in$ exp $\mathscr{P}$. Assume that T is extreme. Let $f_s \geqslant 0$ be such that $\|f_s\| = \|T_s f_s\| = 1$. Then $e_0 = \sum_{s=1}^{s_0} f_s/(s_0)^{1/p}$ is such that $\|e_0\| = \|Te_0\| = 1$ and supp $e_0 = $ supp T. Put $\alpha_j > 0$ such that $e_0 = \sum_{j=1}^n \alpha_j e_j$. Obviously $\alpha_j \leqslant 1$. Suppose, to get a condition, that supp $Te_1 \cap $ supp $Te_2 \neq \varnothing$. Put $\varepsilon > 0$, $E_1, E_2 \subset [0, 1]$ such that $\min(Te_1, Te_2) \geqslant \varepsilon 1_{E_1 \cup E_2}$, $\|1_{E_1}\| = \|1_{E_2}\| \neq 0$, and $E_1 \cap E_2 = \varnothing$. We define an operator $R \in \mathscr{L}(l_n^p, L^p[0, 1])$ by

$$Re_1 = \alpha_2 \beta_2 \varepsilon 1_{E_1} - \alpha_2 \beta_1 \varepsilon 1_{E_2}$$

$$Re_2 = -\alpha_1 \beta_2 \varepsilon 1_{E_1} + \alpha_1 \beta_1 \varepsilon 1_{E_2}$$

$$Re_j = 0 \quad \text{if} \quad j \geqslant 3,$$

where $\beta_i = \langle 1_{E_i}, (Te_0)^{p-1} \rangle$, $i = 1, 2$. Obviously $|\beta_i| = |\langle 1_{E_i}, (Te_i)^{p-1} \rangle| \leqslant \|1_{E_i}\| \|(Te_i)^{p-1}\|_p \leqslant 1$. It is easy to see that $T \pm R \geqslant 0$. We have

$R \neq 0$, $Re_0 = 0$, and $R^*(Te_0)^{p-1} = 0$. Because $(T \pm R) e_0 = Te_0$ and $(T \pm R)^*(Te_0)^{p-1} = e_0^{p-1}$, by Proposition 1 in [16], we have $\|T \pm R\| \leq 1$. Thus T is not extreme. This contradiction shows that supp Te_j are disjoint. Now it is easy to see that $\|Te_j\| = 0$ or 1. Obviously, if $T \in \mathscr{P}$ is such that $\|Te_j\| = 0$ or 1 and supp Te_j are disjoint, then T is extreme.

THEOREM 5. *Let* $0 \neq T \in \mathscr{L}_+(l_n^p, L^r[0,1])$, $n < \infty$, $1 < r \leq p < \infty$. *Then* T *is an extreme positive contraction if and only if* $\|T\| = 1$ *and* supp Te_j, $j = 1, 2, ..., n$, *are disjoint.*

Proof. Suppose that $0 \neq T \in \text{ext } \mathscr{P}$. Then $\|T\| = 1$, and by Theorem 3 in [16], there exists $e_0 = \sum_j \alpha_j e_j \geq 0$ such that $\|e_0\| = \|Te_0\| = 1$. Suppose that supp $Te_1 \cap$ supp $Te_2 \neq \varnothing$. Put $\varepsilon > 0$, $E_1, E_2 \subset [0,1]$, β_1, β_2 such that $\min(Te_1, Te_2) \geq \varepsilon 1_{E_1 \cup E_2}$, $\|1_{E_1}\| = \|1_{E_2}\| \neq 0$, $E_1 \cap E_2 = \varnothing$, $\beta_i = \langle 1_{E_i}, (Te_0)^{r-1} \rangle \leq 1$, $i = 1, 2$. We have $T \pm R \geq 0$ for $R \in \varepsilon(\beta_2 1_{E_1} - \beta_1 1_{E_2}) \otimes (\alpha_2 e_1 - \alpha_1 e_2)$. By Proposition 1 in 1 in [16] we have $\|T \pm R\| \leq 1$, since $Re_0 = 0$ and $R^*(Te_0)^{r-1} = 0$. Thus $T \pm R \in \mathscr{P}$ and $R \neq 0$. This contradiction proves that supp Te_j are disjoint.

Now assume that $\|T\| = 1$ and supp Te_j are disjoint. Suppose that $T \pm R \in \mathscr{P}$ for some $R \in \mathscr{L}(l_n^p, L^r[0,1])$. The sets supp Re_j are disjoint, since supp Te_j are disjoint. By Theorem 3 in [16], there exists $e_0 = \sum_j \alpha_j e_j \geq 0$ such that $\|e_0\| = \|Te_0\| = 1$, supp $e_0 =$ supp T. By the strict convexity of $L^r[0,1]$ we obtain $Re_0 = 0$. Thus $0 = \sum_j \alpha_j Re_j$ and by the disjointness of supp Re_j, we get $Re_j = 0$. Therefore T is extreme.

As a consequence of Theorems 4 and 5 we obtain the following corollaries.

COROLLARY 3. *Let* $T \in \mathscr{L}(L^p[0,1], l_n^p)$, $n < \infty$, $1 < p < \infty$. *Then* T *is an extreme positive contraction if and only if* T *has the form* $T = \sum_{j=1}^n e_j \otimes g_j$ *where* $0 \leq g_j \in L^p[0,1]$ *such that* supp g_j *are disjoint and* $\|g_j\| = 0$ *or 1.*

COROLLARY 4. *Let* $T \in \mathscr{L}(L^p[0,1], l_n^r)$, $n < \infty$, $1 < r < p < \infty$. *Then* T *is an extreme positive contraction if and only if* $\|T\| = 1$ *and* T *has the form* $\sum_{j=1}^n e_j \otimes g_j$, *where* $0 \leq g_j \in L^p[0,1]$ *with disjoint supports.*

Using the same arguments as those from the proof of Theorems 4 and 5 we can obtain the following theorems.

THEOREM 6. *Let* $0 \neq T \in \mathscr{N} \cap \mathscr{L}_+(l^p, L^p[0,1])$ (*or* $T \in \mathscr{N} \cap \mathscr{L}_+(L^p[0,1], l^p)$) $1 < p < \infty$. *Then* $T \in \mathscr{N} \cap \text{ext } \mathscr{P}$ *if and only if* T (*or* T^*) *has the form* $\sum_{j=1}^\infty g_j \otimes e_j$ *where* $g_j \in L^p[0,1]$ *have disjoint supports and* $\|g_j\| = 0$ *or 1.*

THEOREM 7. Let $T \in \mathcal{N} \cap \mathcal{L}_+(l^p, L^r[0, 1])$ (or $T \in \mathcal{N} \cap \mathcal{L}_+(L^p[0, 1], l^r)$) $1 < r < p < \infty$. Then $T \in \mathcal{N} \cap \text{ext}\, \mathcal{P}$ if and only if $\|T\| = 1$ and T (or T^*) has the form $\sum_{j=1}^{\infty} g_1 \otimes e_j$ where $g_j \in L^r_+[0, 1]$ have disjoint supports.

Remark 3. Because a compact operator on the L^p-space is norm attaining, directly from the above characterizations of points of $\mathcal{N} \cap \text{ext}\, \mathcal{P}$, one could obtain corresponding characterizations of extreme points of the positive part of the unit ball of the space of all compact operators. The characterization remains the same as in Theorems 3, 5, and 7.

Now we describe the graph $G(T)$ of all extreme positive contractions in $\mathcal{L}(l^p, l^r)$. The connected components of $G(T)$, $T \in \exp \mathcal{P}$, are trees for $1 < r \leq p < \infty$. It should be pointed out that the form of the graph $G(T)$ does not characterize extreme positive contractions (see Example in [13]).

THEOREM 8. Let T be an extreme positive contraction in $\mathcal{L}(l^p, l^r)$, $1 < r \leq p < \infty$. Then the graph $G(T)$ has no cycle.

Proof. Suppose that $G(T)$ has a simple cycle C. Let I_n denote the projection in $\mathcal{L}(l^p, l^p)$ defined by

$$I_n e_k = \begin{cases} e_k, & \text{if } k \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Let n be sufficiently large (i.e., $G(T_n)$ includes C, where $T_n = TI_n$). Note that T_n are finite dimensional operators and not extreme. It is not difficult to see that for T_n there exists $S_n = (s_{ij}^n)$ such that $\|T_n \pm S_n\| \leq \|T_n\| \leq 1$, $T_n \pm S_n \geq 0$, the graph $G(S_n) = C$, and $t_{i_0 j_0} = |s_{i_0 j_0}^n|$ for some $(i_0, j_0) \in C$. Choose a subsequence $n_k \nearrow \infty$ such that $\lim_{k \to \infty} s_{ij}^{n_k} = s_{ij}^0$ exist for all (i, j). Note that $s_{ij}^0 \neq 0$ for some (i, j), i.e., $S_0(s_{ij}^0) \neq 0$. It is easy to see that

$$\|(T_{n_k} \pm S_{n_k})f - (T \pm S_0)f\| \xrightarrow{k} 0$$

for all $f \in l^p$. Hence $\|T \pm S_0\| \leq 1$. We have $T \pm S_0 \geq 0$. Therefore $T + S_0 \in \mathcal{P}$, i.e., T is not extreme.

Note that the characterization of the extreme positive contractions in $\mathcal{L}(l^p, l^p)$, $1 < p < \infty$, is presented in [12].

5. EXPOSED POINTS

Let Q be a convex set. We recall that $q_0 \in Q$ is exposed if there exists a linear functional ξ such that $\xi(q_0) > \xi(q)$ for all $q \in Q \setminus \{q_0\}$. An exposed point $q_0 \in Q$ is called strongly exposed if for any sequence $q_n \in Q$ the condition $\xi(q_n) \to \xi(q_0)$ implies $q_n \to q_0$. Each exposed point is extreme. Note

that each extreme positive contraction in $\mathscr{L}(l_n^p, l_m^r)$, $n, m < \infty$, $1 < r \leqslant p < \infty$, is strongly exposed (see Theorem 4 in [13]).

THEOREM 9. Let $1 < r \leqslant p < \infty$. Let $T \in \mathscr{L}(l^p, l^r)$. If $T \in \mathscr{N} \cap \text{ext } \mathscr{P}$, then T is exposed but T is not a strongly exposed point of $\mathscr{P}$.

Proof. It is easy to see that $T = 0$ is exposed. Let $0 \neq T \in \mathscr{N} \cap \text{ext } \mathscr{P}$. Denote by $f \in l^p$ such that $\|Tf\| = \|f\| = 1$, $f \geqslant 0$, $\text{supp } f = \text{supp } T$. Put $\varepsilon_{ij} > 0$, $i, j = 1, 2, 3, \ldots$, such that $\sum_{i,j} \varepsilon_{ij} < \infty$. Then T is exposed by a functional ξ defined by

$$\xi(S) = \langle Sf, (Tf)^{r-1} \rangle - \sum_{i,j} \varepsilon_{ij} s_{ij} (1 - \text{sign } t_{ij})$$

$S = (s_{ij}) \in \mathscr{L}(l^p, l^r)$. Indeed, if S is a positive contraction, then $|\xi(S)| \leqslant \|Sf\| \|(Tf)^{r-1}\| \leqslant 1$ and $\xi(T) = 1$.

Now suppose that $\xi(S) = 1$ for some positive contraction $S = (s_{ij})$. Then $Sf = Tf$ and $s_{ij} = 0$ for all (i, j) such that $t_{ij} = 0$. Thus $S \in \mathscr{A}_{f, Tf}$ and the graph $G(S)$ is included in the graph $G(T)$. Because T is uniquely determined by f, Tf, $G(T)$, we obtain $T = S$ (Remark 2). Thus T is exposed.

Now suppose that certain $0 \neq T \in \mathscr{N} \cap \text{ext } \mathscr{P}$ is strongly exposed by functional ξ and $\|T\| = \xi(T) = 1$. In view of Theorem 3 in each (except at most one) column of (t_{ij}) there exist infinitely many zero entries. Therefore there exists a sequence $\{(i_n, j_n)\}_{n=1}^{\infty}$ such that $i_n < i_{n+1}$, $j_n < j_{n+1}$, $t_{i_n j_n} = 0$, $n = 1, 2, 3, \ldots$. Define $R_n = (r_{ij}^n)$ by

$$r_{ij}^n = \begin{cases} t_{ij} & \text{if } i = i_n \\ t_{ij} & \text{if } j = j_n \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $0 \leqslant \sum_{n=1}^{n_0} R_n \leqslant T$ for all $n_0 \in \mathbb{N}$, so $\|\sum_{n=1}^{n_0} R_n\| \leqslant 1$. Thus $\sum_{n=1}^{n_0} \xi(R_n) \leqslant 1$. We have $\xi(R_n) \geqslant 0$, since $T - R_n$ is a positive contraction and $1 - \xi(R_n) = \xi(T - R_n) \leqslant 1$. Therefore $\xi(R_n)$ tends to 0 as n tends to ∞.

Let N_0 be an arbitrary finite subset of $\mathbb{N}$. Because $\sum_{n \in N_0} e_{i_n} \otimes e_{j_n}$ is a positive contraction, we have $\sum_{n \in N_0} \xi(e_{i_n} \otimes e_{j_n}) \leqslant 1$. Thus $\xi(e_{i_n} \otimes e_{j_n})$ tends to 0 as n tends to ∞. Put $T_n = T - R_n + e_{i_n} \otimes e_{j_n}$ tends to 0 as n tends to ∞. Put $T_n = T - R_n + e_{i_n} \otimes e_{j_n}$. We have $T_n \geqslant 0$, $\|T_n\| \leqslant 1$, and $\xi(T_n) = \xi(T) - \xi(R_n) + \xi(e_{i_n} \otimes e_{j_n}) \to^n 1$, but $\|T_n - T\| = \|e_{i_n} \otimes e_{j_n} - R_n\| \geqslant 1$. This contradiction proves that T is not a strongly exposed point of $\mathscr{P}$. Using analogous arguments it is easy to see that $T = 0$ is also not strongly exposed.

The same situation as in Theorem 8 exists when we consider the unit ball of $\mathscr{L}(l^2, l^2)$ (see [14]).

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